Solution 7

1. Determine whether $\mathbb{Z}$ and $\mathbb{Q}$ are complete sets in $\mathbb{R}$.

   Solution. $\mathbb{Z}$ is a closed subset so it is complete. On the other hand, the closure of $\mathbb{Q}$ is $\mathbb{R}$, it is not complete.

2. Does the collection of all differentiable functions on $[a, b]$ form a complete set in $C[a, b]$?

   Solution. No. Since $C[a, b]$ is complete, it suffices to show that the set of differentiable functions is not closed. But this is easy, I leave you to verify the sequence of differentiable functions $f_n(x) = (1/n + x^2)^{1/2}$ in $C[-1, 1]$ converges uniformly to the non-differentiable function $f(x) = |x|$.

3. Let (X, d) be a metric space and $C_b(X)$ the vector space of all bounded, continuous functions in X. Show that it forms a complete metric space under the sup-norm. This problem will be used in the next problem.

   Solution. Let $\{f_n\}$ be a Cauchy sequence in $C_b(X)$. For $\varepsilon > 0$, there exists n_1 such that

   $$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty < \varepsilon, \quad \forall x \in X. \tag{1}$$

   It shows that $\{f_n(x)\}$ is a numerical Cauchy sequence, so $\lim_{n\to\infty} f_n(x)$ exists. We define $f(x) = \lim_{n\to\infty} f_n(x)$. We check it is continuous at x_0 as follows. By passing $m \to \infty$ in (1), we have

   $$|f(x)-f(x_0)| \leq |f(x)-f_{n_1}(x)|+|f_{n_1}(x)-f_{n_1}(x_0)|+|f_{n_1}(x_0)-f(x_0)| \leq 2\varepsilon+|f_{n_1}(x)-f_{n_1}(x_0)|.$$

   As f_{n_1} is continuous, there is some δ such that $|f_{n_1}(x) - f_{n_1}(x_0)| < \varepsilon$ for $x \in B_\delta(x_0)$. It follows that we $|f(x) - f(x_0)| < 3\varepsilon$ for $x \in B_\delta(x_0)$, so f is continuous at x_0. Now, letting $m \to \infty$ in (1), we get $|f_n(x) - f(x)| \leq \varepsilon$ for all $n \geq n_1$, so $f_n \to f$ uniformly. In particular, it means f is bounded.

4. We define a metric on $\mathbb{N}$, the set of all natural numbers by setting

   $$d(n, m) = \left| \frac{1}{n} - \frac{1}{m} \right| .$$

   (a) Show that it is not a complete metric.

   (b) Describe how to make it complete by adding one new point.

   Solution. The sequence $\{n\}$ is a Cauchy sequence in this metric but it has no limit. Its completion is obtained by adding an ideal point called ∞ and define $\tilde{d}(x, y) = d(x, y)$ when $x, y \in \mathbb{Z}$ and $\tilde{d}(x, \infty) = 0$ for all $x \in \mathbb{Z}$ or ∞.

5. Optional. Let (X, d) be a metric space. Fixing a point $p \in X$, for each x define a function

   $$f_x(z) = d(z, x) - d(z, p).$$

   (a) Show that each f_x is a bounded, uniformly continuous function in X.

   (b) Show that the map $x \mapsto f_x$ is an isometric embedding of (X, d) to $C_b(X)$ (shorthand for $C_b(X, \mathbb{R})$) . In other words,

   $$\|f_x - f_y\|_\infty = d(x, y), \quad \forall x, y \in X.$$

   (c) Deduce from (b) the completion theorem.

This approach is much shorter than the proof given in notes. However, it is not so inspiring.

Solution.

(a) From $|f_x(z)| = |d(z,x) - d(z,p)| \le d(x,p)$, and from $|f_x(z) - f_x(z')| \le |d(z,x) - d(z',x)| + |d(z',p) - d(z,p)| \le 2d(z,z')$, it follows that each f_x is a bounded, uniformly continuous function in X.

(b) $|f_x(z) - f_y(z)| = |d(z,x) - d(z,y)| \le d(x,y)$, and equality holds taking $z = x$. Hence

$$\|f_x - f_y\|_\infty = d(x,y), \quad \forall x,y \in X.$$

(c) Let $Y_0 = \{f_x : x \in X\} \subset C_b(X)$. Let Y be the closure of Y_0 in the complete metric space $(C_b(X), \rho)$ with sup-norm ρ. Then (Y, ρ) is a completion of (X, d).

6. Let $f : E \to Y$ be a uniformly continuous map where $E \subset X$ and X, Y are metric spaces. Suppose that Y is complete. Show that there exists a uniformly continuous map F from $\overline{E}$ to Y satisfying $F = f$ in E. In other words, f can be extended to the closure of E preserving uniform continuity.

   Solution. Let $x \in \partial E$. There exists $\{x_n\} \subset E, x_n \to x$. Since $\{x_n\}$ is a Cauchy sequence, by uniformly continuity $\{f(x_n)\}$ is also a Cauchy sequence in Y. As Y is complete, $\{f(x_n)\}$ converges to some point in Y. Therefore, we can define $F(x) = \lim_{n\to\infty} f(x_n)$. It remains to show this definition is independent of the sequence $\{x_n\}$. Indeed, let $\{y_n\}, y_n \to x$. We claim $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} f(x_n)$. It suffices to set $z_{2n+1} = x_n$ when n is odd and $z_{2n} = y_n$ to form a new sequence $\{z_n\}$. This sequence again is a Cauchy sequence, so $\{f(z_n)\}$ is convergent. As both $\{x_n\}$ and $\{y_n\}$ are subsequences of it, $\{f(x_n)\}$ and $\{f(y_n)\}$ tend to the same limit. Now, it is clear that the new function F extends f and is uniformly continuous on the closure of E.

   Note. We have used this property in the proof of Theorem 3.4. Observe that a contraction is always uniformly continuous.

7. Consider maps from $\mathbb{R}$ to itself. Provide explicit examples of continuous maps with exactly one, two and three fixed, and one map satisfying $|f(x) - f(y)| < |x-y|$ but no fixed points.

   Solution. Let f be our function. We consider $g(x) = f(x) - x$. It suffices to produce examples with exactly one, two and three roots. For instance, $g_1(x) = -x$ has exactly one root. $g_2(x) = x^2 - 1$ has exactly two roots. $g_3(x) = (x-1)(x-2)(x-3)$ has exactly three roots. The corresponding f_1, f_2, f_3 fulfil our requirement. Finally, the function $f(x) = x + \log(1 + e^{-x})$ does not have any fixed point.

8. Let T be a continuous map on the complete metric space X. Suppose that for some k, T^k becomes a contraction. Show that T admits a unique fixed point. This generalizes the contraction mapping principle in the case $k = 1$.

   Solution. Since T^k is a contraction, there is a unique fixed point $x \in X$ such that $T^k x = x$. Then $T^{k+1}x = T^k T x = Tx$ shows that Tx is also a fixed point of T^k. From the uniqueness of fixed point we conclude $Tx = x$, that is, x is a fixed point for T. Uniqueness is clear since any fixed point of T is also a fixed point of T^k.

9. Show that the equation $2x\sin x - x^4 + x = 0.001$ has a root near $x = 0$.

Solution. Here $\Psi(x) = 2x\sin x - x^4$. We need to find some r, γ so it is a contraction. We have

$$\begin{aligned} |\Psi(x_1) - \Psi(x_2)| &= \left|2x_1(\sin x_1 - \sin x_2) + 2(x_1 - x_2)\sin x_2 - (x_1^4 - x_2^4)\right| \\ &= \left|2x_1 \cos c(x_1 - x_2) + 2(x_1 - x_2)\sin x_2 - (x_1^2 + x_2^2)(x_1 + x_2)(x_1 - x_2)\right| \\ &\le (2r + r + (2r^2)(2r))|x_1 - x_2| \ . \end{aligned}$$

Taking $r = 1/4, \gamma = 2r + r + (2r^2)(2r) = 13/16 < 1$. By the Perturbation of Identity Theorem, the equation $2x\sin x - x^4 + x = y$ is solvable for any y satisfying $|y| \le R = (1-\gamma)r = 0.0468$, including $y = 0.001$.

10. Can you solve the system of equations

$$x + y^4 = 0, \quad y - x^2 = 0.015 \ ?$$

Solution. Here we work on $\mathbb{R}^2$ and $\Phi(x, y) = (x, y) + \Psi(x, y)$ where $\Psi(x, y) = (-y^4, x^2)$. In the following points in $\mathbb{R}^2$ are denoted by $p = (x_1, y_1), q = (x_2, y_2)$, etc.

$$\begin{aligned} \|\Psi(p) - \Psi(q)\|_2 &= \|(-y_1^4 + y_2^4, x_1^2 - x_2^2)\|_2 \\ &= \|((y_1^2 + y_2^2)(y_1 + y_2)(y_2 - y_1), (x_1 + x_2)(x_1 - x_2)\|_2 \\ &\le \sqrt{(2r^2 \times 2r)^2 + (2r)^2}\|p - q\|_2 \\ &= 2r(1 + 4r^2)\|p - q\|_2 \ . \end{aligned}$$

(We have used $|x_1 - x_2|, |y_1 - y_2| \le \|p - q\|_2$.) Hence by taking $r = 1/4, \gamma = 5/8$ and $R = 3/24 = 0.125$. As $0.015 < 0.125$, the system is solvable.

11. Can you solve the system of equations

$$x + y - x^2 = 0, \quad x - y + xy\sin y = -0.005 \ ?$$

Solution. First we rewrite the system in the form of $I + \Psi$. Indeed, by adding up and subtracting the equations, we see that the system is equivalent to

$$x + (-x^2 + xy\sin y)/2 = -0.0025, \quad y + (-x^2 - xy\sin y)/2 = 0.0025 \ .$$

Now we can take

$$\Psi(x, y) = \frac{1}{2}(-x^2 + xy\sin y, -x^2 - xy\sin y) \ ,$$

and proceed as in the previous problem.